

Chapter 5

Gödel's 1st Incompleteness Theorem

5.1 Gödel Numbers

The idea is the following: intuitively, formulas from arithmetic talk about integers – no matter whether these are standard or not – we can turn them into formulas that talk about the arithmetic itself by encoding formulas, proofs, etc. by integers. This way a formula $\varphi(x)$ may say something like “ x is the code of a closed formula from our language $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ ” or $\psi(x, y)$ may eventually say “ x is the code of a closed formula θ from our language and y is the code of a proof of θ in Robinson arithmetic”.

As always in logic, we start with the terms: given term t we write $\ulcorner t \urcorner$ for its code.

Definition 1.1: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$ -terms

The Gödel numbering of the terms from the language whose signature is $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ is

- $t = 0 \quad \rightsquigarrow \quad \ulcorner t \urcorner = \alpha_3(0, 0, 0)$
- $t = x_n \quad \rightsquigarrow \quad \ulcorner t \urcorner = \alpha_3(n + 1, 0, 0)$
- $t = St_0 \quad \rightsquigarrow \quad \ulcorner t \urcorner = \alpha_3(\ulcorner t_0 \urcorner, 0, 1)$
- $t = t_0 + t_1 \quad \rightsquigarrow \quad \ulcorner t \urcorner = \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 2)$
- $t = t_0 \cdot t_1 \quad \rightsquigarrow \quad \ulcorner t \urcorner = \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 3)$

Lemma 1.1

The set $\mathcal{T}$ of all codes of terms from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{T} = \{ \ulcorner t \urcorner \mid t \text{ is a term from } \mathcal{L}_A \}$$

is *Prim. Rec.*

To almost immediately show this result we need to prove that some stronger form of construction by recursion still produces *Prim. Rec.* functions.

Lemma 1.2

For all *Prim. Rec.* functions $h \in \mathbb{N}^{(\mathbb{N}^{n+p+1})}$, $g \in \mathbb{N}^{(\mathbb{N}^n)}$, $k_1, \dots, k_p \in \mathbb{N}^{\mathbb{N}}$, such that every integer $y > 0$

$$\bigwedge_{0 < i \leq p} k_i(y) < y$$

the function $f \in \mathbb{N}^{(\mathbb{N}^{n+1})}$ defined by

$$(1) \quad f(\vec{x}, 0) = g(\vec{x})$$

$$(2) \quad f(\vec{x}, y + 1) = h(f(\vec{x}, k_1(y)), \dots, f(\vec{x}, k_p(y)), \vec{x}, y, f(\vec{x}, y))$$

is also *Prim. Rec.*

First, notice that the Ackermann function $A \in \mathbb{N}^{(\mathbb{N}^2)}$ defined by

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0, \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0, \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

is not of this form for $A(m, n - 1) < n$ certainly does not hold.

Proof of Lemma 1.1:

Its characteristic function $\chi_{\mathcal{T}} : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by:

$$\text{if } \beta_3^3(k) = 0 \text{ and } \beta_3^2(k) = 0 \text{ and } \beta_3^1(k) = 0 \rightsquigarrow \chi_{\mathcal{T}}(k) = 1$$

$$\text{if } \beta_3^3(k) = 0 \text{ and } \beta_3^2(k) = 0 \text{ and } \beta_3^1(k) > 0 \rightsquigarrow \chi_{\mathcal{T}}(k) = 1$$

$$\text{if } \beta_3^3(k) = 1 \text{ and } \beta_3^2(k) = 0 \text{ and } \beta_3^1(k) > 0 \rightsquigarrow \chi_{\mathcal{T}}(k) = \chi_{\mathcal{T}} \circ \beta_3^1(k)$$

$$\text{if } \beta_3^3(k) = 2 \rightsquigarrow \chi_{\mathcal{T}}(k) = \chi_{\mathcal{T}}(\beta_3^1(k)) \cdot \chi_{\mathcal{T}}(\beta_3^2(k))$$

$$\text{if } \beta_3^3(k) = 3 \rightsquigarrow \chi_{\mathcal{T}}(k) = \chi_{\mathcal{T}}(\beta_3^1(k)) \cdot \chi_{\mathcal{T}}(\beta_3^2(k))$$

$$\text{else} \rightsquigarrow \chi_{\mathcal{T}}(k) = 0.$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function. □

Proof of Lemma 1.2:

The idea of the proof is to define by recursion a function that carries each and everyone of the datas $f(\vec{x}, z)$ for $z < y$ when defining $f(\vec{x}, y)$. To achieve this, we make use of the *Prim. Rec.* functions c and d that were defined in Example 4.1:

- (1) the function $c : \mathbb{N}^{<\omega} \longrightarrow \mathbb{N}$ codes the finite sequences of integers and is defined by

$$\begin{cases} c(\varepsilon) &= 1 \\ c(x_0, \dots, x_p) &= \Pi(0)^{x_0+1} \cdot \Pi(1)^{x_1+1} \dots \Pi(p)^{x_p+1}. \end{cases}$$

- (2) And the function $d \in \mathbb{N}^{(\mathbb{N}^2)}$ that allows, from any integer n , to recover every element of the sequence $\langle x_0, \dots, x_p \rangle$ that c encodes (i.e., $c(x_0, \dots, x_p) = n$) by

$$d(i, n) = \mu x \leq n \quad \Pi(i)^{x+1} \text{ does not divide } n.$$

We want to define some *Prim. Rec.* function $\theta \in \mathbb{N}^{(\mathbb{N}^{n+1})}$ such that

$$\theta(\vec{x}, y) = c(f(\vec{x}, 0), f(\vec{x}, 1), \dots, f(\vec{x}, y))$$

This is easily done by recursion:

- (1) $\theta(\vec{x}, 0) = 2^{g(\vec{x})+1}$
- (2) $\theta(\vec{x}, y+1) = \theta(\vec{x}, y) \cdot \Pi(y+1)^{h[d[k_1(y)), \theta(\vec{x}, y)], \dots, d[k_p(y)), \theta(\vec{x}, y)], \vec{x}, y, f(\vec{x}, y)]+1}.$

Then, to show that f is also *Prim. Rec.*, it only remains to set

$$f(\vec{x}, y) = d[y, \theta(\vec{x}, y)] \div 1.$$

□

Definition 1.2: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$ -formulas

The Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$ -formulas is

$$\begin{array}{llll}
\varphi = t_0 = t_1 & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 4) \\
\varphi = \neg \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \psi \urcorner, 0, 5) \\
\varphi = (\varphi_0 \wedge \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 6) \\
\varphi = (\varphi_0 \vee \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 7) \\
\varphi = (\varphi_0 \longrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 8) \\
\varphi = (\varphi_0 \longleftrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 9) \\
\varphi = \forall x_n \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \psi \urcorner, n, 10) \\
\varphi = \exists x_n \psi & \rightsquigarrow & \ulcorner \varphi \urcorner & = \alpha_3(\ulcorner \psi \urcorner, n, 11).
\end{array}$$

Notice that for every formula φ , we have $\ulcorner \varphi \urcorner > 0$.

Lemma 1.3

The set of all codes of formulas from $\mathcal{L}_A$ is *Prim. Rec.*

$$\mathcal{F} = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a formula from } \mathcal{L}_A\} \in \text{Prim. Rec.}$$

Proof of Lemma 1.3:

Its characteristic function $\chi_{\mathcal{F}} : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{llll}
\text{if } \beta_3^3(k) = 4 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}}(\beta_3^1(k)) \cdot \chi_{\mathcal{F}}(\beta_3^2(k)) \\
\text{if } \beta_3^3(k) = 5 \text{ and } \beta_3^2(k) = 0 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}} \circ \beta_3^1(k) \\
\text{if } \beta_3^3(k) = 6 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}}(\beta_3^1(k)) \cdot \chi_{\mathcal{F}}(\beta_3^2(k)) \\
\text{if } \beta_3^3(k) = 7 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}}(\beta_3^1(k)) \cdot \chi_{\mathcal{F}}(\beta_3^2(k)) \\
\text{if } \beta_3^3(k) = 8 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}}(\beta_3^1(k)) \cdot \chi_{\mathcal{F}}(\beta_3^2(k)) \\
\text{if } \beta_3^3(k) = 9 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}}(\beta_3^1(k)) \cdot \chi_{\mathcal{F}}(\beta_3^2(k)) \\
\text{if } \beta_3^3(k) = 10 \text{ and } \beta_3^2(k) > 0 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}} \circ \beta_3^1(k) \\
\text{if } \beta_3^3(k) = 11 \text{ and } \beta_3^2(k) > 0 & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = \chi_{\mathcal{F}} \circ \beta_3^1(k) \\
\text{else} & \rightsquigarrow & \chi_{\mathcal{F}}(k) & = 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function. □

Lemma 1.4

The occurrence relation of variables in terms from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{T}_{\check{x}} = \{(\ulcorner t \urcorner, n) \mid t \text{ is a term from } \mathcal{L}_{\mathcal{A}} \text{ and } t \text{ contains } x_n\}$$

is *Prim. Rec.*

Proof of Lemma 1.4:

Its characteristic function $\chi_{\mathcal{T}_{\check{x}}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{aligned} \text{if } \beta_3^3(k) &= 0 \text{ and } \beta_3^2(k) = 0 \text{ and } \beta_3^1(k) = n+1 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}}}(k, n) = 1 \\ \text{if } \beta_3^3(k) &= 1 \text{ and } \beta_3^2(k) = 0 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}}}(k, n) = \chi_{\mathcal{T}_{\check{x}_n}}(\beta_3^1(k), n) \\ \text{if } \beta_3^3(k) &= 2 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}}}(k, n) = \max(\chi_{\mathcal{T}_{\check{x}_n}}(\beta_3^1(k), n), \chi_{\mathcal{T}_{\check{x}_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) &= 3 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}}}(k, n) = \max(\chi_{\mathcal{T}_{\check{x}_n}}(\beta_3^1(k), n), \chi_{\mathcal{T}_{\check{x}_n}}(\beta_3^2(k), n)) \\ \text{else} &&& \rightsquigarrow \chi_{\mathcal{T}_{\check{x}}}(k, n) = 0. \end{aligned}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function. □

Lemma 1.5

The set

$$\mathcal{T}_{\check{x}x} = \{(\ulcorner t \urcorner, n) \mid t \text{ is a term from } \mathcal{L}_{\mathcal{A}} \text{ and } t \text{ does not contain } x_n\}$$

is *Prim. Rec.*

Proof of Lemma 1.5:

Its characteristic function $\chi_{\mathcal{T}_{\check{x}x_n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by :

$$\begin{aligned} \text{if } \beta_3^3(k) &= 0 \text{ and } \beta_3^2(k) = 0 \text{ and } \beta_3^1(k) \neq n+1 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}x_n}}(k, n) = 1 \\ \text{if } \beta_3^3(k) &= 1 \text{ and } \beta_3^2(k) = 0 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}x_n}}(k, n) = \chi_{\mathcal{T}_{\check{x}x_n}}(\beta_3^1(k), n) \\ \text{if } \beta_3^3(k) &= 2 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}x_n}}(k, n) = \chi_{\mathcal{T}_{\check{x}x_n}}(\beta_3^1(k), n) \cdot \chi_{\mathcal{T}_{\check{x}x_n}}(\beta_3^2(k), n) \\ \text{if } \beta_3^3(k) &= 3 && \rightsquigarrow \chi_{\mathcal{T}_{\check{x}x_n}}(k, n) = \chi_{\mathcal{T}_{\check{x}x_n}}(\beta_3^1(k), n) \cdot \chi_{\mathcal{T}_{\check{x}x_n}}(\beta_3^2(k), n) \\ \text{else} &&& \rightsquigarrow \chi_{\mathcal{T}_{\check{x}x_n}}(k, n) = 0. \end{aligned}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function.

□

Lemma 1.6

The set

$$\mathcal{F}_{\checkmark x} = \{(\ulcorner \varphi \urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathcal{A}} \text{ and } \varphi \text{ contains } x_n\}$$

is *Prim. Rec.*

Proof of Lemma 1.6:

Its characteristic function $\chi_{\mathcal{F}_{\checkmark x}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{ll} \text{if } \beta_3^3(k) = 4 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \max(\chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^1(k), n), \chi_{\mathcal{T}_{\checkmark x_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) = 5 \text{ and } \beta_3^2(k) = 0 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n) \\ \text{if } \beta_3^3(k) = 6 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \max(\chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) = 7 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \max(\chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) = 8 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \max(\chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) = 9 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \max(\chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^2(k), n)) \\ \text{if } \beta_3^3(k) = 10 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n) \\ \text{if } \beta_3^3(k) = 11 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = \chi_{\mathcal{F}_{\checkmark x_n}}(\beta_3^1(k), n) \\ \text{else} & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x}}(k, n) = 0. \end{array}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function.

□

Lemma 1.7

The set

$$\mathcal{F}_{\times x} = \{(\ulcorner \varphi \urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathcal{A}} \text{ and } \varphi \text{ does not contain } x_n\}$$

is *Prim. Rec.*

Proof of Lemma 1.7:

Its characteristic function $\chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by :

$$\begin{array}{ll}
 \text{if } \beta_3^3(k) = 4 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{T}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{T}_{\textcolor{red}{x}x_n}} \circ \beta_3^2(k) \\
 \text{if } \beta_3^3(k) = 5 \quad \text{and} \quad \beta_3^2(k) = 0 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \\
 \text{if } \beta_3^3(k) = 6 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^2(k) \\
 \text{if } \beta_3^3(k) = 7 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^2(k) \\
 \text{if } \beta_3^3(k) = 8 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^2(k) \\
 \text{if } \beta_3^3(k) = 9 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \cdot \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^2(k) \\
 \text{if } \beta_3^3(k) = 10 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \\
 \text{if } \beta_3^3(k) = 11 & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}} \circ \beta_3^1(k) \\
 \text{else} & \rightsquigarrow \chi_{\mathcal{F}_{\textcolor{red}{x}x_n}}(k, n) = 0.
 \end{array}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function. □

Lemma 1.8

The set

$$\mathcal{F}_{\textcolor{green}{x} \text{ free}} = \{(\textcolor{blue}{\ulcorner} \varphi \urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_A \text{ and } x_n \text{ is free in } \varphi\}$$

is *Prim. Rec.*

Proof of Lemma 1.8:

Its characteristic function $\chi_{\mathcal{F}_{\textcolor{green}{x} \text{ free}}} : \mathbb{N}^2 \longrightarrow \mathbb{N}$ is defined by:

$$\begin{array}{ll}
\text{if } \beta_3^3(k) = 4 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \max\left(\chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^2(k), n)\right) \\
\text{if } \beta_3^3(k) = 5 \text{ and } \beta_3^2(k) = 0 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n) \\
\text{if } \beta_3^3(k) = 6 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \max\left(\chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^2(k), n)\right) \\
\text{if } \beta_3^3(k) = 7 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \max\left(\chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^2(k), n)\right) \\
\text{if } \beta_3^3(k) = 8 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \max\left(\chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^2(k), n)\right) \\
\text{if } \beta_3^3(k) = 9 & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \max\left(\chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n), \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^2(k), n)\right) \\
\text{if } \beta_3^3(k) = 10 \text{ and } \beta_3^2(k) \neq n & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n) \\
\text{if } \beta_3^3(k) = 11 \text{ and } \beta_3^2(k) \neq n & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(\beta_3^1(k), n) \\
\text{else} & \rightsquigarrow \chi_{\mathcal{F}_{\checkmark x \text{ free}}}(k, n) = 0.
\end{array}$$

By Lemma 1.2 this definition by case study yields a *Prim. Rec.* function. $\square$

Lemma 1.9

The set

$$\mathcal{F}_{\checkmark x \text{ bound}} = \{(\ulcorner \varphi \urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathcal{A}} \text{ and } x_n \text{ is bound in } \varphi\}$$

is *Prim. Rec.*

Proof of Lemma 1.9:

we have $\mathcal{F}_{\checkmark x \text{ bound}} = \mathcal{F}_{\checkmark x} \setminus \mathcal{F}_{\checkmark x \text{ free}}$. $\square$

Lemma 1.10

The set of all codes of closed formulas from $\mathcal{L}_{\mathcal{A}}$

$$\mathcal{F}_{\checkmark \text{ closed}} = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a closed formula from } \mathcal{L}_{\mathcal{A}}\}$$

is *Prim. Rec.*

Proof of Lemma 1.10:

We have

$$k \in \mathcal{F}_{\checkmark \text{closed}} \iff k \in \mathcal{F} \text{ and } \forall n \leq k \ (k, n) \notin \mathcal{F}_{\checkmark x \text{ free}}.$$

□

Lemma 1.11

The function $\mathcal{S}_{ub.}^{\mathcal{T}} \in \mathbb{N}^{(\mathbb{N}^3)}$ defined below is *Prim. Rec.*

$$\mathcal{S}_{ub.}^{\mathcal{T}}(n_u, n_t, n) = \begin{cases} \ulcorner u[t/x_n] \urcorner & \text{if } n_u \in \mathcal{T}, n_t \in \mathcal{T} \text{ and } n_u = \ulcorner u \urcorner, n_t = \ulcorner t \urcorner \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 1.11:

We first recall the definition of $\ulcorner t \urcorner$:

$$\begin{aligned} t &= 0 & \rightsquigarrow & \ulcorner t \urcorner = \alpha_3(0, 0, 0) \\ t &= x_n & \rightsquigarrow & \ulcorner t \urcorner = \alpha_3(n+1, 0, 0) \\ t &= St_0 & \rightsquigarrow & \ulcorner t \urcorner = \alpha_3(\ulcorner t \urcorner, 0, 1) \\ t &= t_0 + t_1 & \rightsquigarrow & \ulcorner t \urcorner = \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 2) \\ t &= t_0 \cdot t_1 & \rightsquigarrow & \ulcorner t \urcorner = \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 3) \end{aligned}$$

$\mathcal{S}_{ub.}^{\mathcal{T}} \in \mathbb{N}^{(\mathbb{N}^3)}$ is defined by

$$\mathcal{S}_{ub.}^{\mathcal{T}}(n_u, n_t, n) = \begin{cases} 0 & \text{if } n_u \notin \mathcal{T} \text{ or } n_t \notin \mathcal{T} \\ n_t & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 0 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) = n+1 \\ n_u & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 0 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) \neq n+1 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^1(n_u), n_t, n), 0, 1) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 1 \text{ and } \beta_3^2(n_u) = 0 \text{ and } \beta_3^1(n_u) \in \mathcal{T} \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^1(n_u), n_t, n), \mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^2(n_u), n_t, n), 2) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 2 \text{ and } \beta_3^2(n_u) \in \mathcal{T} \text{ and } \beta_3^1(n_u) \in \mathcal{T} \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^1(n_u), n_t, n), \mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^2(n_u), n_t, n), 3) & \text{if } n_u, n_t \in \mathcal{T} \text{ and } \beta_3^3(n_u) = 3 \text{ and } \beta_3^2(n_u) \in \mathcal{T} \text{ and } \beta_3^1(n_u) \in \mathcal{T}. \end{cases}$$

By Lemma 1.2 $\mathcal{S}_{ub.}^{\mathcal{T}}$ is *Prim. Rec.*

□

Lemma 1.12

The function $\mathcal{S}_{ub.}^{\mathcal{F}} \in \mathbb{N}^{(\mathbb{N}^3)}$ defined below is *Prim. Rec.*

$$\mathcal{S}_{ub.}^{\mathcal{F}}(n_{\varphi}, n_t, n) = \begin{cases} \ulcorner \varphi[t/x_n] \urcorner & \text{if } n_{\varphi} = \ulcorner \varphi \urcorner \in \mathcal{F}, n_t = \ulcorner t \urcorner \in \mathcal{T} \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Lemma 1.12:

We first recall the definition of $\ulcorner \varphi \urcorner$:

$$\begin{array}{llll} \varphi = t_0 = t_1 & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner t_0 \urcorner, \ulcorner t_1 \urcorner, 4) \\ \varphi = \neg \psi & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \psi \urcorner, 0, 5) \\ \varphi = (\varphi_0 \wedge \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 6) \\ \varphi = (\varphi_0 \vee \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 7) \\ \varphi = (\varphi_0 \longrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 8) \\ \varphi = (\varphi_0 \longleftrightarrow \varphi_1) & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \varphi_0 \urcorner, \ulcorner \varphi_1 \urcorner, 9) \\ \varphi = \forall x_n \psi & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \psi \urcorner, n, 10) \\ \varphi = \exists x_n \psi & \rightsquigarrow & \ulcorner \varphi \urcorner = & \alpha_3(\ulcorner \psi \urcorner, n, 11). \end{array}$$

$\mathcal{S}_{ub.}^{\mathcal{F}} \in \mathbb{N}^{(\mathbb{N}^3)}$ is defined by

$$\mathcal{S}_{ub.}^{\mathcal{F}}(n_{\varphi}, n_t, n) = \begin{cases} 0 & \text{if } n_{\varphi} \notin \mathcal{F} \text{ or } n_t \notin \mathcal{T} \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^1(n_{\varphi}), n_t, n), \mathcal{S}_{ub.}^{\mathcal{T}}(\beta_3^2(n_{\varphi}), n_t, n), 4) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 4 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), 0, 5) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 5 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^2(n_{\varphi}), n_t, n), 6) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 6 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^2(n_{\varphi}), n_t, n), 7) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 7 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^2(n_{\varphi}), n_t, n), 8) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 8 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^2(n_{\varphi}), n_t, n), 9) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 9 \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \beta_3^2(n_{\varphi}), 10) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 10 \text{ and } \beta_3^2(n_{\varphi}) \neq n \\ \alpha_3(\mathcal{S}_{ub.}^{\mathcal{F}}(\beta_3^1(n_{\varphi}), n_t, n), \beta_3^2(n_{\varphi}), 11) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 11 \text{ and } \beta_3^2(n_{\varphi}) \neq n \\ \beta_3^1(n_{\varphi}) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 10 \text{ and } \beta_3^2(n_{\varphi}) = n \\ \beta_3^1(n_{\varphi}) & \text{if } n_{\varphi} \in \mathcal{F} \text{ and } n_t \in \mathcal{T} \text{ and } \beta_3^3(n_{\varphi}) = 11 \text{ and } \beta_3^2(n_{\varphi}) = n \end{cases}$$

By Lemma 1.2 $\mathcal{S}_{ub.}^{\mathcal{F}}$ is *Prim. Rec.*

□

We will now define a way of coding (finite) sets of formulas. We will not really encode the set,

but some finite sequence of formulas, because we will not care about the ordering of such a sequence, even if what we really encode is the sequence, we will handle it as if it were a set.

Definition 1.3: Coding and decoding sequences

We define both $\ulcorner \cdot \urcorner : \mathbb{N}^{<\omega} \longrightarrow \mathbb{N}$ and $\lfloor \cdot \rfloor : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by

$$\begin{cases} \ulcorner \varepsilon \urcorner &= 0 \\ \ulcorner k_0, \dots, k_p \urcorner &= \Pi(0)^{k_0} \cdot \Pi(1)^{k_1} \dots \Pi(p)^{k_p}. \end{cases}$$

Where $\Pi(i)$ enumerates the prime numbers^a.

And

$$\lfloor n \rfloor^i = \mu x \leq n \quad \Pi(i)^{x+1} \text{ does not divide } n.$$

^a $\Pi(0) = 2; \Pi(1) = 3; \Pi(2) = 5$; etc.

Notice that for all $i \leq p$ we have $\lfloor \ulcorner k_0, \dots, k_p \urcorner \rfloor^i = k_i$. Furthermore, for every formula φ , the integer $\ulcorner \varphi \urcorner$ is strictly positive. Therefore, given any sequence $\langle k_0, \dots, k_p \rangle \in \mathbb{N}^{<\omega}$ if $\ulcorner k_0, \dots, k_p \urcorner = 0$ then we know for sure that k_i does not code a formula.

We will say that the integer 1 codes the empty set – which is also an empty set of formulas – and another integer codes the set $\Delta = \{\varphi_0, \varphi_1, \dots, \varphi_p\}$ if this integer is of the form $\Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \dots \Pi(i_p)^{\ulcorner \varphi_p \urcorner}$.

Definition 1.4: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$ -finite sets of formulas

The Gödel numbering of any set $\Delta = \{\varphi_0, \varphi_1, \dots, \varphi_p\}$ of $\mathcal{L}_{\mathcal{A}}$ -formulas is any integer of the form

$$\begin{aligned} \ulcorner \Delta \urcorner &= 1 && \text{if } \Delta = \emptyset, \\ &= \Pi(i_0)^{\ulcorner \varphi_0 \urcorner} \cdot \Pi(i_1)^{\ulcorner \varphi_1 \urcorner} \dots \Pi(i_p)^{\ulcorner \varphi_p \urcorner} && \text{otherwise.} \end{aligned}$$

with $\{i_0, \dots, i_p\}$ and Δ having the same cardinality^a.

We denote $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ the set of codes of finite sets of formulas:

$$\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} = \{\ulcorner \Delta \urcorner \mid \Delta \text{ is any finite set of } \mathcal{L}_{\mathcal{A}} \text{ formulas}\}.$$

^aThis means $\forall j, k \leq p \quad (j \neq k \rightarrow i_j \neq i_k)$.

Lemma 1.13

The set $\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}$ of codes of finite sets of formulas is *Prim. Rec.*

Proof of Lemma 1.13:

$$\chi_{\mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n \neq 1 \text{ and } \forall i \leq n \quad [\ulcorner n^i \urcorner > 0 \rightarrow \ulcorner n^i \urcorner \in \mathcal{F}] \\ 0 & \text{else.} \end{cases}$$

□

Lemma 1.14

There exist two *Prim. Rec.* functions $\mathcal{R}_{em.} : \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\mathcal{A}_{dd.} : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that

$$\begin{aligned} \mathcal{A}_{dd.}(n, m) &= \begin{cases} \ulcorner \Delta \cup \{\varphi\} \urcorner & \text{if } n = \ulcorner \varphi \urcorner \in \mathcal{F} \text{ and } m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \text{if } n \notin \mathcal{F} \text{ or } m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \end{cases} \\ \mathcal{R}_{em.}(n, m) &= \begin{cases} \ulcorner \Delta \setminus \{\varphi\} \urcorner & \text{if } n = \ulcorner \varphi \urcorner \in \mathcal{F} \text{ and } m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \text{if } n \notin \mathcal{F} \text{ or } m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}. \end{cases} \end{aligned}$$

Proof of Lemma 1.14:

We have both

$$\mathcal{A}_{dd.}(n, m) = \begin{cases} 0 & \text{if } n \notin \mathcal{F} \text{ or } m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ m \cdot \prod (\mu i \leq m \quad \ulcorner m^i \urcorner = 0)^n & \text{if } \ulcorner \varphi \urcorner = n \in \mathcal{F} \text{ and } \ulcorner \Delta \urcorner = m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}. \end{cases}$$

and

$$\mathcal{R}_{em.}(n, m) = \begin{cases} 0 & \text{if } n \notin \mathcal{F} \text{ or } m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ m & \text{if } n \in \mathcal{F} \text{ and } m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \forall i \leq m \quad \ulcorner m^i \urcorner \neq n \\ \left[\frac{m}{\prod (\mu i \leq m \quad \ulcorner m^i \urcorner = n)^n} \right] & \text{if } n \in \mathcal{F} \text{ and } m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \exists i \leq m \quad \ulcorner m^i \urcorner = n \end{cases}$$

□

Lemma 1.15

The set

$$\mathcal{I}_{ns.} = \left\{ (\ulcorner \varphi \urcorner, \ulcorner \Delta \urcorner) \in \mathbb{N}^2 \mid \ulcorner \varphi \urcorner \in \mathcal{F}, \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \varphi \in \Delta \right\}$$

is *Prim. Rec.*

Later on, we will use a relation-like notation and write $\ulcorner \varphi \urcorner \mathcal{I}_{ns.} \ulcorner \Delta \urcorner$ instead of $(\ulcorner \varphi \urcorner, \ulcorner \Delta \urcorner) \in \mathcal{I}_{ns.}$.

Proof of Lemma 1.15:

We have

$$\chi_{\mathcal{I}_{ns.}}(n, m) = \begin{cases} 1 & \text{if } n \in \mathcal{F} \text{ and } m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \exists i \leq m \begin{cases} \Pi(i)^n \text{ divides } m \\ \text{and} \\ \Pi(i)^{n+1} \text{ does not divides } m \end{cases} \\ 0 & \text{else.} \end{cases}$$

□

Lemma 1.16

The following set

$$\mathcal{E}_{qu.} = \left\{ (\ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner) \in \mathbb{N}^2 \mid \ulcorner \Gamma \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})}, \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \Gamma = \Delta \right\}.$$

is *Prim. Rec.*

Notice that the equality “ $\Gamma = \Delta$ ” is between two sets, therefore it relies on extensionality. We will use a relation-like notation and write $\ulcorner \varphi \urcorner \mathcal{E}_{qu.} \ulcorner \Delta \urcorner$ instead of $(\ulcorner \varphi \urcorner, \ulcorner \Delta \urcorner) \in \mathcal{E}_{qu.}$.

Proof of Lemma 1.16:

We have

$$\chi_{\mathcal{E}_{qu.}}(n, m) = \begin{cases} 1 & \text{if } n, m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \forall i \leq \max(n, m) \ (i \in \mathcal{F} \rightarrow (i \mathcal{I}_{ns.} n \leftrightarrow i \mathcal{I}_{ns.} m)) \\ 0 & \text{else.} \end{cases}$$

□

Lemma 1.17

There exists a *Prim. Rec.* function $\mathcal{U}_{nion} \in \mathbb{N}^{(\mathbb{N}^2)}$ such that

$$\mathcal{U}_{nion}(n, m) = \begin{cases} 0 & \text{if } n \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ or } m \notin \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ \ulcorner \Theta \urcorner & \text{if } n = \ulcorner \Gamma \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } m = \ulcorner \Delta \urcorner \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } \ulcorner \Theta \urcorner \mathcal{E}_{qu.} \ulcorner \Gamma \cup \Delta \urcorner. \end{cases}$$

Proof of Lemma 1.17:

We first define $f \in \mathbb{N}^{(\mathbb{N}^3)}$ by recursion:

$$\begin{cases} f(\gamma, \delta, 0) &= \Pi(0)^{\ulcorner \gamma^0 \urcorner} \cdot \Pi(1)^{\ulcorner \delta^0 \urcorner} \\ f(\gamma, \delta, n+1) &= \Pi(2n)^{\ulcorner \gamma^n \urcorner} \cdot \Pi(2n+1)^{\ulcorner \delta^n \urcorner} \cdot f(\gamma, \delta, n). \end{cases}$$

Then we set

$$\mathcal{U}_{nion}(\gamma, \delta) = f(\gamma, \delta, \max(\gamma, \delta)).$$

$$\begin{aligned} \mathcal{U}_{nion}(n, m) &= f(n, m, \max(n, m)) \quad \text{if } n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \text{ and } m \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

□

Definition 1.5: Gödel numbering of the $\mathcal{L}_{\mathcal{A}}$ -sequents from *sequent calculus*

The Gödel numbering of any sequent $\Gamma \vdash \Delta$ is

$$\ulcorner \Gamma \vdash \Delta \urcorner = \alpha_2(\ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner)$$

We denote $\mathcal{SQ}$ the set of codes of sequents:

$$\mathcal{SQ} = \{\ulcorner \Gamma \vdash \Delta \urcorner \mid \Gamma, \Delta \text{ finite sets of } \mathcal{L}_{\mathcal{A}} \text{ formulas}\}.$$

Given any integer n we use the notation ${}^l n$ for $\beta_2^1(n)$ and ${}^r n$ for $\beta_2^2(n)$. This way,

$$\text{if } n = \ulcorner \Gamma \vdash \Delta \urcorner, \text{ then } {}^l n = \ulcorner \Gamma \urcorner \text{ and } {}^r n = \ulcorner \Delta \urcorner.$$

Lemma 1.18

The set $\mathcal{SQ}$ of codes of sequents of *sequent calculus* is *Prim. Rec.*

Proof of Lemma 1.18:

$$\chi_{\mathcal{SQ}}(n) = \begin{cases} 1 & \text{if } {}^l n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \quad \text{and} \quad {}^r n \in \mathcal{C}_{\mathcal{P}_{fin.}(\mathcal{F})} \\ 0 & \text{else} \end{cases}$$

□

We will now denote $\mathcal{AX}$ the set of codes of axioms of sequent calculus which are not to be mistaken for the axiom of Robinson arithmetic.

Definition 1.6: Gödel numbering of the axioms of *sequent calculus*

$$\mathcal{AX} = \left\{ \alpha_2(2^{\ulcorner \varphi \urcorner}, 2^{\ulcorner \varphi \urcorner}) \mid \ulcorner \varphi \urcorner \in \mathcal{F} \right\}.$$

Lemma 1.19

The set $\mathcal{AX}$ of codes of axioms of *sequent calculus* is *Prim. Rec.*

Proof of Lemma 1.19:

$$\chi_{\mathcal{AX}}(n) = \begin{cases} 1 & \text{if } {}^l n = {}^r n \text{ and } {}^l n^{\ulcorner 0 \urcorner} \in \mathcal{F} \text{ and } \forall i \leq n \quad {}^l n^{\ulcorner i+1 \urcorner} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

□

5.2 Coding the Proofs

We recall that a proof in Sequent Calculus is a tree of the form

$$\begin{array}{c}
\frac{\overline{\psi \vdash \psi}^{ax}}{\varphi, \psi \vdash \psi}^{wkn_l} \quad \frac{\overline{\varphi \vdash \varphi}^{ax}}{\varphi \vdash \varphi, \psi}^{wkn_r} \\
\frac{\varphi, \psi \vdash \psi}{\varphi \vdash \neg\psi, \psi}^{\neg_r} \quad \frac{\varphi \vdash \varphi, \psi}{\varphi, \neg\varphi \vdash \psi}^{\neg_l} \\
\frac{\varphi, \neg\psi \rightarrow \neg\varphi \vdash \psi}{\neg\psi \rightarrow \neg\varphi \vdash \varphi \rightarrow \psi}^{\rightarrow_l}
\end{array}$$

where the shape of the tree is controlled by the rules of Sequent Calculus.

We are now ready to define for each rule of the Sequent Calculus, a set of tuples of codes of sequents that satisfy the property that the rule defines.

We will successively define

Definition 2.1

- (1) $\circ \mathcal{R}_{ax} \subseteq \mathbb{N}$
- (2) $\circ \mathcal{R}_{\wedge_{l1}} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\exists_l} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\forall_r} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{ctr_{l\&r}} \subseteq \mathbb{N}^2$
 $\circ \mathcal{R}_{\wedge_{l2}} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\vee_{r1}} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\exists_r} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{cut} \subseteq \mathbb{N}^2$
 $\circ \mathcal{R}_{\neg_l} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\vee_{r2}} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{wkn_l} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{Rep} \subseteq \mathbb{N}^2$
 $\circ \mathcal{R}_{\forall_l} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{\neg_r} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{wkn_r} \subseteq \mathbb{N}^2$ $\circ \mathcal{R}_{Ref} \subseteq \mathbb{N}^2$
- (3) $\circ \mathcal{R}_{\vee_l} \subseteq \mathbb{N}^3$ $\circ \mathcal{R}_{\rightarrow_l} \subseteq \mathbb{N}^3$ $\circ \mathcal{R}_{\wedge_r} \subseteq \mathbb{N}^3$

and for each of them, the fact that it is *Prim. Rec.* will derive from its definition. We first recall what the rules are.

Sequent Calculus

Axioms

$$\frac{}{\varphi \vdash \varphi} \text{ ax}$$

Logical Rules

$$\begin{array}{c} \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \wedge_{l1} \quad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \wedge_{l2} \quad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta} \wedge_r \\[10pt] \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \vee_l \quad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \vee_{r1} \quad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \vee_{r2} \\[10pt] \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \rightarrow \psi \vdash \Delta} \rightarrow_l \quad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \rightarrow \psi, \Delta} \rightarrow_r \\[10pt] \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg \varphi \vdash \Delta} \neg_l \quad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg \varphi, \Delta} \neg_r \\[10pt] \frac{\Gamma, \varphi[t/x] \vdash \Delta^1}{\Gamma, \forall x \varphi \vdash \Delta} \forall_l \quad \frac{\Gamma \vdash \varphi[y/x], \Delta}{\Gamma \vdash \forall x \varphi, \Delta^2} \forall_r \\[10pt] \frac{\Gamma, \varphi[y/x] \vdash \Delta}{\Gamma, \exists x \varphi \vdash \Delta^2} \exists_l \quad \frac{\Gamma \vdash \varphi[t/x], \Delta^1}{\Gamma \vdash \exists x \varphi, \Delta} \exists_r \\[10pt] \frac{\Gamma, t=t \vdash \Delta}{\Gamma \vdash \Delta} \text{Ref} \quad \frac{\Gamma, t=s, \varphi[s/x], \varphi[t/x] \vdash \Delta}{\Gamma, s=t, \varphi[t/x] \vdash \Delta} \text{Rep} \end{array}$$

Structural Rules

$$\begin{array}{c} \frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \text{wkn}_l \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta} \text{wkn}_r \\[10pt] \frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \text{ctr}_l \quad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta} \text{ctr}_r \end{array}$$

Cut Rule

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{cut}$$

¹for t a term

²for y a variable with no free occurrence in the sequent concluding the rule (not in $\Gamma, \exists x \varphi$ nor $\forall x \varphi$, nor Δ)

$$\boxed{\frac{}{\varphi \vdash \varphi}^{ax}}$$

$$D \in \mathcal{R}_{ax} \iff D \in \mathcal{AX}$$

.....

$$\boxed{\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}^{\wedge_{l1}}}$$

$$(U, D) \in \mathcal{R}_{\wedge_{l1}}$$

$$\iff$$

$$\left\{ \begin{array}{l} U \in \mathcal{SQ} \\ \text{and} \\ D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu}. {}^rD \\ \text{and} \\ \exists {}^r\varphi \leqslant {}^lU \exists {}^r\psi \leqslant {}^lD \end{array} \right. \left(\begin{array}{l} {}^r\varphi \mathcal{I}_{ns}. {}^lU \quad \text{and} \quad {}^r\varphi \wedge \psi \mathcal{I}_{ns}. {}^lD \\ \text{and} \\ \mathcal{R}_{em.}({}^r\varphi, {}^lU) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\varphi \wedge \psi, {}^lD) \end{array} \right)$$

where

- “ ${}^r\varphi \wedge \psi$ ” stands for “ $\alpha_3({}^r\varphi, {}^r\psi, 6)$ ”.
- “ $\exists {}^r\varphi \leqslant k \quad \theta_{[{}^r\varphi/y]}$ ” stands for “ $\exists n \leqslant k \quad (n \in \mathcal{F} \wedge \theta_{[n/y]})$ ” and more generally
- “ $\exists {}^r\varphi_1 \leqslant k_1 \dots \dots \exists {}^r\varphi_n \leqslant k_n \quad \theta_{[{}^r\varphi_1/y_1, \dots, {}^r\varphi_n/y_n]}$ ” stands for
“ $\exists p \leqslant \alpha_n(k_1, \dots, k_n) \quad \left(\bigwedge_{i \leqslant n} (\beta_n^i(p) \in \mathcal{F} \wedge \beta_n^i(p) \leqslant k_i) \quad \wedge \quad \theta_{[\beta_n^1(p)/y_1, \dots, \beta_n^n(p)/y_n]} \right)$ ”.

.....

$$\boxed{\frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \wedge_{I2}}$$

$$(U, D) \in \mathcal{R}_{\wedge_{I2}}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U \in \mathcal{SQ} \\ \text{and} \\ D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^r\psi \leq {}^lU \exists {}^r\varphi \leq {}^lD \end{array} \right. \left(\begin{array}{l} {}^r\psi \mathcal{I}_{ns.} {}^lU \quad \text{and} \quad {}^r\varphi \wedge \psi \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.}({}^r\psi, {}^lU) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\varphi \wedge \psi, {}^lD) \end{array} \right)$$

.....

$$\boxed{\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \vee_I}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\vee_I}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U_l, U_r, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU_l \mathcal{E}_{qu.} {}^rU_r \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^r\varphi \leq {}^lU_l \exists {}^r\psi \leq {}^lU_r \end{array} \right. \left(\begin{array}{l} {}^r\varphi \mathcal{I}_{ns.} {}^lU_l \quad \text{and} \quad {}^r\psi \mathcal{I}_{ns.} {}^lU_r \quad \text{and} \quad {}^r\varphi \wedge \psi \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.}({}^r\varphi, {}^lU_l) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\psi, {}^lU_r) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\varphi \wedge \psi, {}^lD) \end{array} \right)$$

.....

$$\boxed{\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \rightarrow \psi \vdash \Delta} \rightarrow_i}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\rightarrow_l}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U_l, U_r, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU_r \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^r\varphi \leqslant {}^rU_l \exists {}^r\psi \leqslant {}^lU_r \end{array} \right. \left(\begin{array}{l} {}^r\varphi \mathcal{I}_{ns.} {}^rU_l \text{ and } {}^r\psi \mathcal{I}_{ns.} {}^lU_r \text{ and } {}^r\varphi \wedge {}^r\psi \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.} ({}^r\varphi, {}^rU_l) \mathcal{E}_{qu.} {}^rU_r \\ \text{and} \\ \mathcal{R}_{em.} ({}^r\psi, {}^lU_r) \mathcal{E}_{qu.} {}^lU_l \mathcal{E}_{qu.} \mathcal{R}_{em.} ({}^r\varphi \rightarrow {}^r\psi, {}^lD) \\ \text{and} \\ \mathcal{R}_{em.} ({}^r\psi, {}^lU_r) \mathcal{E}_{qu.} \mathcal{R}_{em.} ({}^r\varphi \rightarrow {}^r\psi, {}^lD) \end{array} \right)$$

◦ where “ $A \mathcal{E}_{qu.} B \mathcal{E}_{qu.} C$ ” stands for “ $A \mathcal{E}_{qu.} B$ and $B \mathcal{E}_{qu.} C$ ”

.....

$$\boxed{\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg \varphi \vdash \Delta} \neg_i}$$

$$(U, D) \in \mathcal{R}_{\neg_l}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ \exists {}^r\varphi \leqslant {}^rU \end{array} \right. \left(\begin{array}{l} {}^r\varphi \mathcal{I}_{ns.} {}^rU \text{ and } {}^r\neg \varphi \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.} ({}^r\varphi, {}^rU) \mathcal{E}_{qu.} \mathcal{R}_{em.} ({}^r\neg \varphi, {}^lD) \end{array} \right)$$

.....

$$\boxed{\frac{\Gamma, \varphi[t/x_n] \vdash \Delta}{\Gamma, \forall x_n \varphi \vdash \Delta} \forall_l}$$

$$(U, D) \in \mathcal{R}_{\forall_l}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists n \leqslant {}^lD \quad \exists {}^{\textcolor{blue}{\varphi}}\forall x_n \varphi \leqslant {}^lD \quad \exists {}^{\textcolor{teal}{t}} \leqslant {}^lU \end{array} \right. \left(\begin{array}{l} \left(\begin{array}{l} {}^{\textcolor{blue}{\forall x_n \varphi}} \mathcal{I}_{ns.} {}^lD \quad \text{and} \quad ({}^{\textcolor{blue}{\varphi}}, n) \in \mathcal{F}_{\checkmark x \text{ free}} \\ \text{and} \\ \mathcal{S}_{ub.}^{\mathcal{F}}({}^{\textcolor{blue}{\varphi}}, {}^{\textcolor{teal}{t}}, n) \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ \mathcal{R}_{em.}(\mathcal{S}_{ub.}^{\mathcal{F}}({}^{\textcolor{blue}{\varphi}}, {}^{\textcolor{teal}{t}}, n), {}^lU) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^{\textcolor{blue}{\forall x_n \varphi}}, {}^lD) \end{array} \right) \\ \text{or} \\ \left(\begin{array}{l} {}^{\textcolor{blue}{\forall x_n \varphi}} \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ ({}^{\textcolor{blue}{\varphi}}, n) \in (\mathcal{F}_{\textcolor{red}{x}} \cup \mathcal{F}_{\checkmark x \text{ bound}}) \\ \text{and} \\ {}^{\textcolor{blue}{\varphi}} \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ \mathcal{S}_{ub.}^{\mathcal{F}}({}^{\textcolor{blue}{\varphi}}, {}^{\textcolor{teal}{t}}, n) \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ \mathcal{R}_{em.}({}^{\textcolor{blue}{\varphi}}, {}^lU) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^{\textcolor{blue}{\forall x_n \varphi}}, {}^lD) \end{array} \right) \end{array} \right)$$

○ “ $\exists n \leqslant {}^rU \quad \exists {}^{\textcolor{blue}{\forall x_n \varphi}} \leqslant {}^rU \dots$ ” stands for

“ $\exists n \leqslant {}^rU \quad \exists m \leqslant {}^rU \quad \left(m \in \mathcal{F} \wedge \beta_3^3(m) = 10 \wedge \beta_3^2(m) = n \wedge \beta_3^1(m) = {}^{\textcolor{blue}{\varphi}} \wedge \dots \right)$ ”

○ “ ${}^{\textcolor{blue}{\varphi}}$ ” stands for “ $\beta_3^1({}^{\textcolor{blue}{\forall x_n \varphi}})$ ”

○ “ $\exists {}^{\textcolor{teal}{t}} \leqslant {}^lU \dots$ ” stands for “ $\exists v \leqslant {}^lU (v \in \mathcal{T} \wedge \dots)$ ”

.....

$$\boxed{\frac{\Gamma, \varphi[x_k/x_n] \vdash \Delta}{\Gamma, \exists x_n \varphi \vdash \Delta^2} \exists_l}$$

$$(U, D) \in \mathcal{R}_{\exists_l}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^r x_n \leqslant {}^l D \quad \exists {}^r x_k \leqslant {}^l U \quad \exists {}^r \exists x_n \varphi \leqslant {}^l D \quad \exists {}^r \psi \leqslant {}^l U \end{array} \right. \left(\begin{array}{l} \left(\begin{array}{l} {}^r \exists x_n \varphi \mathcal{I}_{ns.} {}^l D \quad \text{and} \quad {}^r \psi \mathcal{I}_{ns.} {}^l U \\ \text{and} \\ ({}^r \psi, k) \in \mathcal{F}_{\check{x} \text{ free}} \quad \text{and} \quad (k \neq n \rightarrow ({}^r \psi, n) \in (\mathcal{F}_{\check{x}} \cup \mathcal{F}_{\check{x} \text{ bound}})) \\ \text{and} \\ \alpha_3(\mathcal{S}_{ub.}({}^r \psi, {}^r x_n, k), n, 11) = {}^r \exists x_n \varphi \\ \text{and} \\ \mathcal{R}_{em.}({}^r \psi, {}^l U) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r \exists x_n \varphi, {}^l D) \\ \text{and} \\ \forall {}^r \theta \leqslant {}^l D \quad ({}^r \theta \mathcal{I}_{ns.} {}^l D \rightarrow ({}^r \theta, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}})) \\ \text{and} \\ \forall {}^r \delta \leqslant {}^r D \quad ({}^r \delta \mathcal{I}_{ns.} {}^r D \rightarrow ({}^r \delta, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}})) \end{array} \right) \\ \text{or} \\ \left(\begin{array}{l} {}^r \exists x_n \varphi \mathcal{I}_{ns.} {}^l D \quad \text{and} \quad {}^r \psi \mathcal{I}_{ns.} {}^l U \quad \text{and} \\ ({}^r \psi, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}}) \quad \text{and} \quad ({}^r \psi, n) \in (\mathcal{F}_{\check{x}} \cup \mathcal{F}_{\check{x} \text{ bound}}) \\ \text{and} \quad \alpha_3({}^r \psi, n, 11) = {}^r \exists x_n \varphi \quad \text{and} \\ \mathcal{R}_{em.}({}^r \psi, {}^l U) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r \exists x_n \varphi, {}^l D) \end{array} \right) \end{array} \right)$$

where

- “ $\exists n \leqslant {}^r U \quad \exists {}^r \forall x_n \varphi \leqslant {}^r U \dots$ ” stands for
“ $\exists n \leqslant {}^r U \quad \exists m \leqslant {}^r U \quad \left(m \in \mathcal{F} \wedge \beta_3^3(m) = 11 \wedge \beta_3^2(m) = n \wedge \beta_3^1(m) = {}^r \varphi \wedge \dots \right)$ ”
- “ ${}^r \varphi$ ” stands for “ $\beta_3^1({}^r \forall x_n \varphi)$ ”
- “ $\exists {}^r x_k \leqslant {}^l U \dots$ ” stands for “ $\exists k \leqslant {}^l U \quad ({}^r x_k = \alpha_3(k+1, 0, 0) \wedge \dots)$ ”

.....

² x_k has no free occurrence in $\Gamma, \exists x_n \varphi$ and Δ

$$\boxed{\frac{\Gamma, t = t \vdash \Delta}{\Gamma \vdash \Delta} \text{Ref}}$$

$$(U, D) \in \mathcal{R}_{Ref}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^{\textcolor{teal}{r}}t \leqslant {}^lU \left(\begin{array}{c} {}^{\textcolor{blue}{r}}t = {}^{\textcolor{teal}{r}}t \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ \mathcal{R}_{em.} ({}^{\textcolor{blue}{r}}t = {}^{\textcolor{teal}{r}}t, {}^lU) \mathcal{E}_{qu.} {}^lD \end{array} \right) \end{array} \right.$$

.....

$$\boxed{\frac{\Gamma, t = s, \varphi[s/x_n], \varphi[t/x_n] \vdash \Delta}{\Gamma, s = t, \varphi[t/x_n] \vdash \Delta} \text{Rep}}$$

$$(U, D) \in \mathcal{R}_{Rep}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists {}^{\textcolor{teal}{r}}t \leqslant {}^lU \exists {}^{\textcolor{teal}{r}}s \leqslant {}^lU \exists n \leqslant U \exists {}^{\textcolor{blue}{r}}\varphi \leqslant U^U \left(\begin{array}{l} {}^{\textcolor{blue}{r}}t = {}^{\textcolor{blue}{r}}s \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ \mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}s, n) \mathcal{I}_{ns.} {}^lU \text{ and } \mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}t, n) \mathcal{I}_{ns.} {}^lU \\ \text{and} \\ {}^{\textcolor{blue}{r}}s = {}^{\textcolor{teal}{r}}t \mathcal{I}_{ns.} {}^lD \text{ and } \mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}t, n) \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.} (\mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}t, n), \mathcal{R}_{em.} ({}^{\textcolor{blue}{r}}s = {}^{\textcolor{teal}{r}}t, {}^lD)) \\ \mathcal{E}_{qu.} \\ \mathcal{R}_{em.} (\mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}s, n), \mathcal{R}_{em.} (\mathcal{S}_{ub.}^{\mathcal{F}} ({}^{\textcolor{blue}{r}}\varphi, {}^{\textcolor{teal}{r}}t, n), \mathcal{R}_{em.} ({}^{\textcolor{blue}{r}}t = {}^{\textcolor{blue}{r}}s, {}^lU))) \end{array} \right) \end{array} \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta} \wedge_r}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\wedge_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U_l, U_r, D \in \mathcal{SQ} \\ \text{and} \\ {}^l U_l \mathcal{E}_{qu.} {}^l U_r \mathcal{E}_{qu.} {}^l D \\ \text{and} \\ \exists {}^r \varphi \leqslant {}^r U_l \exists {}^r \psi \leqslant {}^r U_r \end{array} \left(\begin{array}{l} {}^r \varphi \mathcal{I}_{ns.} {}^r U_l \text{ and } {}^r \psi \mathcal{I}_{ns.} {}^r U_r \text{ and } {}^r \varphi \wedge {}^r \psi \mathcal{I}_{ns.} {}^r D \\ \text{and} \\ \mathcal{R}_{em.}({}^r \varphi, {}^r U_l) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r \psi, {}^r U_r) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r \varphi \wedge {}^r \psi, {}^r D) \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \vee_{r1}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r1}}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^l U \mathcal{E}_{qu.} {}^l D \\ \text{and} \\ \exists {}^r \varphi \leqslant {}^r U \exists {}^r \psi \leqslant {}^r D \end{array} \left(\begin{array}{l} {}^r \varphi \mathcal{I}_{ns.} {}^r U \text{ and } {}^r \varphi \vee {}^r \psi \mathcal{I}_{ns.} {}^r D \\ \text{and} \\ \mathcal{R}_{em.}({}^r \varphi, {}^r U) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r \varphi \vee {}^r \psi, {}^r D) \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \vee_{r2}}$$

$$(U, D) \in \mathcal{R}_{\vee_{r2}}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^lU \mathcal{E}_{qu.} {}^lD \\ \text{and} \\ \exists {}^r\psi \leqslant {}^rU \exists {}^r\varphi \leqslant {}^rD \end{array} \left(\begin{array}{l} {}^r\psi \mathcal{I}_{ns.} {}^rU \text{ and } {}^r\varphi \vee \psi \mathcal{I}_{ns.} {}^rD \\ \text{and} \\ \mathcal{R}_{em.}({}^r\psi, {}^rU) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\varphi \vee \psi, {}^rD) \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \rightarrow \psi, \Delta} \rightarrow_r}$$

$$(U_l, U_r, D) \in \mathcal{R}_{\rightarrow_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U_l, U_r, D \in \mathcal{SQ} \\ \text{and} \\ \exists {}^r\varphi \leqslant {}^lU_l \exists {}^r\psi \leqslant {}^rU_r \end{array} \left(\begin{array}{l} {}^r\varphi \mathcal{I}_{ns.} {}^lU_l \text{ and } {}^r\psi \mathcal{I}_{ns.} {}^rU_r \text{ and } {}^r\varphi \rightarrow \psi \mathcal{I}_{ns.} {}^rD \\ \text{and} \\ \mathcal{R}_{em.}({}^r\psi, {}^rU_r) \mathcal{E}_{qu.} \mathcal{R}_{em.}({}^r\varphi \rightarrow \psi, {}^rD) \\ \text{and} \\ \mathcal{R}_{em.}({}^r\varphi, {}^lU_l) \mathcal{E}_{qu.} {}^lD \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg \varphi, \Delta} \neg_r}$$

$$(U, D) \in \mathcal{R}_{\neg_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ \exists \textcolor{blue}{\ulcorner \varphi \urcorner} \leqslant {}^l U \end{array} \left(\begin{array}{l} \textcolor{blue}{\ulcorner \varphi \urcorner} \mathcal{I}_{ns.} {}^l U \quad \text{and} \quad \textcolor{blue}{\ulcorner \neg \varphi \urcorner} \mathcal{I}_{ns.} {}^r D \\ \text{and} \\ \mathcal{R}_{em.} (\textcolor{blue}{\ulcorner \varphi \urcorner}, {}^l U) \quad \mathcal{E}_{qu.} {}^l D \\ \text{and} \\ \mathcal{R}_{em.} (\textcolor{blue}{\ulcorner \neg \varphi \urcorner}, {}^r D) \quad \mathcal{E}_{qu.} {}^r U \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \varphi[x_k/x_n], \Delta}{\Gamma \vdash \forall x_n \varphi, \Delta^2} \forall_r}$$

$$(U, D) \in \mathcal{R}_{\forall_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^lU \mathcal{E}_{qu}. {}^lD \\ \text{and} \\ \exists {}^r x_n \leq {}^r D \quad \exists {}^r x_k \leq {}^r U \quad \exists {}^r \forall x_n \varphi \leq {}^r D \quad \exists {}^r \psi \leq {}^r U \end{array} \right. \left(\begin{array}{l} \left(\begin{array}{l} {}^r \forall x_n \varphi \mathcal{I}_{ns}. {}^r D \quad \text{and} \quad {}^r \psi \mathcal{I}_{ns}. {}^r U \\ \text{and} \\ ({}^r \psi, k) \in \mathcal{F}_{\check{x} \text{ free}} \quad \text{and} \quad (k \neq n \rightarrow ({}^r \psi, n) \in (\mathcal{F}_{\check{x}} \cup \mathcal{F}_{\check{x} \text{ bound}})) \\ \text{and} \\ \alpha_3(\mathcal{S}_{ub}.({}^r \psi, {}^r x_n, k), n, 11) = {}^r \forall x_n \varphi \\ \text{and} \\ \mathcal{R}_{em}.({}^r \psi, {}^r U) \mathcal{E}_{qu}. \mathcal{R}_{em}.({}^r \forall x_n \varphi, {}^r D) \\ \text{and} \\ \forall {}^r \theta \leq {}^r D \quad ({}^r \theta \mathcal{I}_{ns}. {}^r D \rightarrow ({}^r \theta, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}})) \\ \text{and} \\ \forall {}^r \delta \leq {}^r D \quad ({}^r \delta \mathcal{I}_{ns}. {}^r D \rightarrow ({}^r \delta, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}})) \end{array} \right) \\ \text{or} \\ \left(\begin{array}{l} {}^r \forall x_n \varphi \mathcal{I}_{ns}. {}^r D \quad \text{and} \quad {}^r \psi \mathcal{I}_{ns}. {}^r U \quad \text{and} \\ ({}^r \psi, k) \in (\mathcal{F}_{\check{x}_k} \cup \mathcal{F}_{\check{x}_k \text{ bound}}) \quad \text{and} \quad ({}^r \psi, n) \in (\mathcal{F}_{\check{x}} \cup \mathcal{F}_{\check{x} \text{ bound}}) \\ \text{and} \quad \alpha_3({}^r \psi, n, 11) = {}^r \forall x_n \varphi \quad \text{and} \\ \mathcal{R}_{em}.({}^r \psi, {}^r U) \mathcal{E}_{qu}. \mathcal{R}_{em}.({}^r \forall x_n \varphi, {}^r D) \end{array} \right) \end{array} \right)$$

where

- “ $\exists n \leq {}^r U \quad \exists {}^r \forall x_n \varphi \leq {}^r U \dots$ ” stands for
“ $\exists n \leq {}^r U \quad \exists m \leq {}^r U \quad \left(m \in \mathcal{F} \wedge \beta_3^3(m) = 10 \wedge \beta_3^2(m) = n \wedge \beta_3^1(m) = {}^r \varphi \wedge \dots \right)$ ”
- “ ${}^r \varphi$ ” stands for “ $\beta_3^1({}^r \forall x_n \varphi)$ ”
- “ $\exists {}^r x_k \leq {}^r U \dots$ ” stands for “ $\exists k \leq {}^r U \left({}^r x_k = \alpha_3(k+1, 0, 0) \wedge \dots \right)$ ”

.....

² x_k has no free occurrence in Γ , $\forall x_n \varphi$ and Δ

$$\boxed{\frac{\Gamma \vdash \varphi[t/x_n], \Delta}{\Gamma \vdash \exists x_n \varphi, \Delta} \exists_r}$$

$$(U, D) \in \mathcal{R}_{\exists_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^lU \mathcal{E}_{qu}. {}^lD \\ \text{and} \\ \exists n \leqslant {}^rD \quad \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^rD \quad \exists \ulcorner t \urcorner \leqslant {}^rU \end{array} \right. \left(\begin{array}{l} \left(\begin{array}{l} \ulcorner \exists x_n \varphi \urcorner \mathcal{I}_{ns}. {}^rD \quad \text{and} \quad (\ulcorner \varphi \urcorner, n) \in \mathcal{F}_{\check{x} \text{ free}} \\ \text{and} \\ \mathcal{S}_{ub}^{\mathcal{F}}(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n) \mathcal{I}_{ns}. {}^rU \\ \text{and} \\ \mathcal{R}_{em.}(\mathcal{S}_{ub}^{\mathcal{F}}(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n), {}^rU) \mathcal{E}_{qu}. \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^rD) \end{array} \right) \\ \text{or} \\ \left(\begin{array}{l} \ulcorner \exists x_n \varphi \urcorner \mathcal{I}_{ns}. {}^rD \\ \text{and} \\ (\ulcorner \varphi \urcorner, n) \in (\mathcal{F}_{\check{x}} \cup \mathcal{F}_{\check{x} \text{ bound}}) \\ \text{and} \\ \ulcorner \varphi \urcorner \mathcal{I}_{ns}. {}^rU \\ \text{and} \\ \mathcal{S}_{ub}^{\mathcal{F}}(\ulcorner \varphi \urcorner, \ulcorner t \urcorner, n) \mathcal{I}_{ns}. {}^rU \\ \text{and} \\ \mathcal{R}_{em.}(\ulcorner \varphi \urcorner, {}^rU) \mathcal{E}_{qu}. \mathcal{R}_{em.}(\ulcorner \exists x_n \varphi \urcorner, {}^rD) \end{array} \right) \end{array} \right)$$

○ “ $\exists n \leqslant {}^rU \quad \exists \ulcorner \exists x_n \varphi \urcorner \leqslant {}^rU \dots$ ” stands for

“ $\exists n \leqslant {}^rU \quad \exists m \leqslant {}^rU \quad \left(m \in \mathcal{F} \wedge \beta_3^3(m) = 11 \wedge \beta_3^2(m) = n \wedge \beta_3^1(m) = \ulcorner \varphi \urcorner \wedge \dots \right)$ ”

○ “ $\ulcorner \varphi \urcorner$ ” stands for “ $\beta_3^1(\ulcorner \exists x_n \varphi \urcorner)$ ”

○ “ $\exists \ulcorner t \urcorner \leqslant {}^rU \dots$ ” stands for “ $\exists v \leqslant {}^rU (v \in \mathcal{T} \wedge \dots)$ ”

.....

$$\boxed{\frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \text{ } wkn_l}$$

$$(U, D) \in \mathcal{R}_{wkn_l}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^rU \mathcal{E}_{qu.} {}^rD \\ \text{and} \\ \exists \textcolor{blue}{\ulcorner \varphi \urcorner} \leqslant {}^lD \end{array} \left(\begin{array}{c} \textcolor{blue}{\ulcorner \varphi \urcorner} \mathcal{I}_{ns.} {}^lD \\ \text{and} \\ \mathcal{R}_{em.} (\textcolor{blue}{\ulcorner \varphi \urcorner}, {}^lD) \mathcal{E}_{qu.} {}^lU \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \Delta}{\Gamma \vdash \varphi, \Delta} \text{ } wkn_r}$$

$$(U, D) \in \mathcal{R}_{wkn_r}$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^lU \mathcal{E}_{qu.} {}^lD \\ \text{and} \\ \exists \textcolor{blue}{\ulcorner \varphi \urcorner} \leqslant {}^rD \end{array} \left(\begin{array}{c} \textcolor{blue}{\ulcorner \varphi \urcorner} \mathcal{I}_{ns.} {}^rD \\ \text{and} \\ \mathcal{R}_{em.} (\textcolor{blue}{\ulcorner \varphi \urcorner}, {}^rD) \mathcal{E}_{qu.} {}^rU \end{array} \right) \right.$$

.....

$$\boxed{\frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \text{ctr}_l} \qquad \boxed{\frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta} \text{ctr}_r}$$

$$(U, D) \in \mathcal{R}_{ctr\ l\&r} \iff \left\{ \begin{array}{l} U, D \in \mathcal{SQ} \\ \text{and} \\ {}^lU \ \mathcal{E}_{qu.} \ {}^lD \\ \text{and} \\ {}^rU \ \mathcal{E}_{qu.} \ {}^rD \end{array} \right.$$

.....

$$\boxed{\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{cut}}$$

$$(U_l, U_r, D) \in \mathcal{R}_{cut}$$

$$\iff$$

$$\left\{ \begin{array}{l} U_l, U_r, D \in \mathcal{SQ} \\ \text{and} \\ \exists \textcolor{blue}{\ulcorner \varphi \urcorner} \leqslant {}^rU_l \quad \left(\begin{array}{l} \textcolor{blue}{\ulcorner \varphi \urcorner} \mathcal{I}_{ns.} \ {}^rU_l \ \text{and} \ \textcolor{blue}{\ulcorner \varphi \urcorner} \mathcal{I}_{ns.} \ {}^lU_r \\ \text{and} \\ \mathcal{U}_{nion}(\mathcal{R}_{em.}(\textcolor{blue}{\ulcorner \varphi \urcorner}, {}^lU_r), {}^lU_l) \ \mathcal{E}_{qu.} \ {}^lD \\ \text{and} \\ \mathcal{U}_{nion}(\mathcal{R}_{em.}(\textcolor{blue}{\ulcorner \varphi \urcorner}, {}^rU_l), {}^rU_r) \ \mathcal{E}_{qu.} \ {}^rD \end{array} \right) \end{array} \right.$$

.....

We use the following notations:

Notation 2.1

- $\mathcal{R}^0 = \mathcal{R}_{ax}$
- $\mathcal{R}^1 = \left\{ \begin{array}{l} \mathcal{R}_{\wedge_{l1}} \cup \mathcal{R}_{\wedge_{l2}} \cup \mathcal{R}_{\neg_l} \cup \mathcal{R}_{\forall_l} \cup \mathcal{R}_{\exists_l} \cup \mathcal{R}_{\vee_{r1}} \cup \mathcal{R}_{\vee_{r2}} \cup \mathcal{R}_{\neg_r} \cup \mathcal{R}_{\forall_r} \\ \cup \mathcal{R}_{\exists_r} \cup \mathcal{R}_{wkn_l} \cup \mathcal{R}_{wkn_r} \cup \mathcal{R}_{ctr_{l\&r}} \cup \mathcal{R}_{Rep} \cup \mathcal{R}_{Ref} \cup \mathcal{R}_{cut} \end{array} \right.$
- $\mathcal{R}^2 = \mathcal{R}_{\vee_l} \cup \mathcal{R}_{\rightarrow_l} \cup \mathcal{R}_{\wedge_r}$.

We say an integer codes a proof if it is of the form

$$\alpha_4(\text{root}, \text{left proof-tree}, \text{right proof-tree}, \text{arity of the rule}).$$

Definition 2.2

The set $\mathcal{P}_{roofs}$ of the codes of all possible proofs is defined by

$$k = \alpha_4(n_1, n_2, n_3, n_4) \in \mathcal{P}_{roofs} \iff \left\{ \begin{array}{l} n_4 = 0 \quad \text{and} \quad n_3 = 0 \quad \text{and} \quad n_2 = 0 \quad \text{and} \quad n_1 \in \mathcal{R}^0 \\ \text{or} \\ n_4 = 1 \quad \text{and} \quad n_3 = 0 \quad \text{and} \quad n_2 \in \mathcal{P}_{roofs} \quad \text{and} \quad (\beta_4^1(n_2), n_1) \in \mathcal{R}^1 \\ \text{or} \\ n_4 = 2 \quad \text{and} \quad n_3 \in \mathcal{P}_{roofs} \quad \text{and} \quad n_2 \in \mathcal{P}_{roofs} \quad \text{and} \quad (\beta_4^1(n_3), \beta_4^1(n_2), n_1) \in \mathcal{R}^2. \end{array} \right.$$

Notation 2.2

Given any proof P we write $\textcolor{teal}{P}$ for the integer described above that codes this proof.

Lemma 2.1

The set $\mathcal{P}_{roofs}$ is *Prim. Rec.*

Proof of Lemma 2.1:

$$\chi_{\mathcal{P}_{roofs}}(n) = \begin{cases} 1 & \text{if } \beta_4^4(n) = 0 \text{ and } \beta_4^3(n) = 0 \text{ and } \beta_4^2(n) = 0 \text{ and } \beta_4^1(n) \in \mathcal{R}^0 \\ 1 & \text{if } \beta_4^4(n) = 1 \text{ and } \beta_4^3(n) = 0 \text{ and } \chi_{\mathcal{P}_{roofs}}(\beta_4^2(n)) = 1 \text{ and } (\beta_4^1 \circ \beta_4^2(n), \beta_4^1(n)) \in \mathcal{R}^1 \\ 1 & \text{if } \beta_4^4(n) = 2 \text{ and } \chi_{\mathcal{P}_{roofs}}(\beta_4^3(n)) = \chi_{\mathcal{P}_{roofs}}(\beta_4^2(n)) = 1 \text{ and } (\beta_4^1 \circ \beta_4^3(n), \beta_4^1 \circ \beta_4^2(n), \beta_4^1(n)) \in \mathcal{R}^2 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1.2, $\mathcal{P}_{roofs}$ is *Prim. Rec.*

□

5.3 Undecidability of Robinson Arithmetic

Since Robinson Arithmetic is some very weak theory, one might think that it should be easy to solve any question posed in such a theory. To the contrary, it turns out that Robinson Arithmetic is undecidable.

Definition 3.1

- (1) A theory T is recursive if the following set is recursive:

$$\left\{ \ulcorner \varphi \urcorner \mid \varphi \in T \right\}.$$

- (2) A theory T is decidable if the following set is recursive:

$$thms(T) = \left\{ \ulcorner \varphi \urcorner \mid T \vdash_c \varphi \right\}.$$

Informally, this means that a theory is decidable if one has an algorithm (in fact, a *Decider*) which on any input that represents a formula φ stops and accepts if T proves φ , and stops and rejects if T does not prove φ .

Theorem 3.1

Given any $\mathcal{L}_A$ -theory T , the set

$$\left\{ (\ulcorner P \urcorner, \ulcorner \varphi \urcorner) \in \mathbb{N}^2 \mid P \text{ is a proof of } T \vdash_c \varphi \right\}$$

is

- primitive recursive if T is primitive recursive,
- recursive if T is recursive.

Proof of Theorem 3.1:

First, P is proof that $T \vdash_c \varphi$ if P is a proof-tree whose root is some sequent “ $\Delta \vdash \varphi$ ” for some finite $\Delta \subseteq T$.

We recall Lemma 1.10 which stated that the following set is *Prim. Rec.*

$$\mathcal{F}_{\checkmark}^{\text{closed}} = \{ \ulcorner \varphi \urcorner \mid \varphi \text{ is a closed formula from } \mathcal{L}_A \}$$

Second, let $\chi_T \in \mathbb{N} \longrightarrow \mathbb{N}$ be the characteristic function of T . i.e.,

$$\chi_T(n) = \begin{cases} 1 & \text{if } n = \ulcorner \varphi \urcorner \in T \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of

$$A = \left\{ (\ulcorner P \urcorner, \ulcorner \varphi \urcorner) \in \mathbb{N}^2 \mid P \text{ is a proof of } T \vdash_c \varphi \right\}$$

is

$$\chi_A(n, m) = \begin{cases} 1 & \text{if} \\ 0 & \text{otherwise.} \end{cases} \left\{ \begin{array}{l} n \in \mathcal{P}_{\text{proofs}} \\ \text{and} \\ m \in \mathcal{F} \\ \text{and} \\ \forall i \leq {}^l\beta_4^1(n) \quad ({}^l\beta_4^1(n)^{\lceil i} \neq 0 \longrightarrow \chi_T({}^l\beta_4^1(n)^{\lceil i}) = 1) \\ \text{and} \\ \forall j \leq {}^r\beta_4^1(n) \quad ({}^r\beta_4^1(n)^{\lceil j} = m \vee {}^r\beta_4^1(n)^{\lceil j} = 0) \\ \text{and} \\ \exists j \leq {}^r\beta_4^1(n) \quad ({}^r\beta_4^1(n)^{\lceil j} = m) \end{array} \right.$$

We see that this function is primitive recursive if χ_T is primitive recursive, and total recursive if χ_T is total recursive. □

Proposition 3.1

Given any $\mathcal{L}_A$ -theory T ,

$$\left\{ \lceil \psi \rceil \mid \psi \in T \right\} \text{ is recursive } \implies \left\{ \lceil \varphi \rceil \mid T \vdash_c \varphi \right\} \text{ is recursively enumerable.}$$

Proof of Proposition 3.1:

We set

$$A = \left\{ (\lceil P \rceil, \lceil \varphi \rceil) \in \mathbb{N}^2 \mid P \text{ is a proof of } T \vdash_c \varphi \right\} \text{ and } B = \left\{ \lceil \varphi \rceil \mid T \vdash_c \varphi \right\}.$$

By Theorem 3.1, the set A is recursive. Hence the function

$$\chi_B^{\text{part}}(n) = 1 \div \left(1 \div (\mu k \quad \chi_A(k, n) = 1) \right)$$

is also *Part. Rec.* and it satisfies $n \in B \iff \chi_B^{\text{part}}(n) = 1$. □

We recall that a theory is **complete** if it satisfies both

- (1) $T \not\vdash_c \perp$ (i.e., T is *consistent*)
 (2) for every formula φ , either $T \vdash_c \varphi$ or $T \vdash_c \neg\varphi$ holds

Corollary 3.1

Let T be any recursive $\mathcal{L}_{\mathcal{A}}$ -theory.

If T is complete, then T is decidable.

Proof of Corollary 3.1:

By Proposition 3.1 both sets

$$\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\} \quad \text{and} \quad \{\ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi\}$$

are recursively enumerable. Since T is complete we have

$$\{\ulcorner\varphi\urcorner \mid T \not\vdash_c \varphi\} = \{\ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi\}$$

Hence

$$\mathbb{N} \setminus \{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\} = (\mathbb{N} \setminus \mathcal{F}) \cup \{\ulcorner\varphi\urcorner \mid T \vdash_c \neg\varphi\}$$

is recursively enumerable, which yields the result. $\square$

Now we come to the main result of the chapter. Gödel's first incompleteness theorem will be an immediate consequence of this undecidability result.

Theorem 3.2: Undecidability of any consistent recursive extension of $\mathcal{R}ob$.

Let $T \supseteq \mathcal{R}ob$. be any $\mathcal{L}_{\mathcal{A}}$ -theory.

If T is consistent, then T is undecidable.

Notice that Theorem 3.2 could also be stated:

Let T be any $\mathcal{L}_{\mathcal{A}}$ -theory which extends $\mathcal{R}ob$.

$$T \text{ is consistent} \iff T \text{ is undecidable.}$$

Because if a theory T is inconsistent, then it proves everything, therefore we have

$$\{\ulcorner\varphi\urcorner \mid T \vdash_c \varphi\} = \mathcal{F}.$$

which makes T decidable.

Proof of Theorem 3.2:

Towards a contradiction, we assume that T is decidable (i.e., $\{\ulcorner \psi \urcorner \mid T \vdash_c \psi\}$ is recursive). We then consider

$$\mathcal{F}_{\checkmark x_0 \text{ !free}} = \{\ulcorner \varphi \urcorner \mid \varphi \text{ is a formula whose only free variable is } x_0\}.$$

Since we already know that the set

$$\mathcal{F}_{\checkmark x \text{ free}} = \{(\ulcorner \varphi \urcorner, n) \mid \varphi \text{ is a formula from } \mathcal{L}_A \text{ and } x_n \text{ is free in } \varphi\}$$

is *Prim. Rec.* (see Lemma 1.8) and we have

$$\ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \iff (\ulcorner \varphi \urcorner, 0) \in \mathcal{F}_{\checkmark x \text{ free}} \quad \text{and} \quad \forall n \leq \ulcorner \varphi \urcorner \quad (n \neq 0 \rightarrow (\ulcorner \varphi \urcorner, n) \notin \mathcal{F}_{\checkmark x \text{ free}}).$$

an immediate consequence is that $\mathcal{F}_{\checkmark x_0 \text{ !free}}$ is also *Prim. Rec.*

Then the set^a

$$\begin{aligned} & \{(\ulcorner \varphi \urcorner, n) \mid \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \text{ and } T \vdash_c \varphi[n/x_0]\} \\ &= \\ & \{(\ulcorner \varphi \urcorner, n) \mid \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \text{ and } \mathcal{S}_{ub}^{\mathcal{F}}(\ulcorner \varphi \urcorner, \ulcorner n \urcorner, 0) \in \{\ulcorner \psi \urcorner \mid T \vdash_c \psi\}\} \end{aligned}$$

is recursive.

We then consider the following set

$$\mathcal{D}_{diag.}^{\ddot{}} = \left\{ k \in \mathbb{N} \mid (k, k) \notin \left\{ (\ulcorner \varphi \urcorner, n) \mid \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \text{ and } T \vdash_c \varphi[n/x_0] \right\} \right\}.$$

i.e. for any integer k we have:

$$k \notin \mathcal{D}_{diag.}^{\ddot{}} \iff k = \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \text{ and } T \vdash_c \varphi[\ulcorner \varphi \urcorner / x_0],$$

which is exactly

$$k \notin \mathcal{D}_{diag.}^{\ddot{}} \iff k = \ulcorner \varphi \urcorner \in \mathcal{F}_{\checkmark x_0 \text{ !free}} \text{ and } T \vdash_c \varphi[\underbrace{\ulcorner S \cdots S \urcorner}_{\ulcorner \varphi \urcorner} 0 / x_0].$$

$\mathcal{D}_{diag.}^{\ddot{}}$ is clearly recursive. Therefore, by Theorem 2.2, there exists some formula $\varphi_{\ddot{}}(x_0)$ (with x_0 being the only free variable) that represents $\mathcal{D}_{diag.}^{\ddot{}}$.

This means that for all $k \in \mathbb{N}$ we have:

- $k \in \mathcal{D}_{diag}^{\ddot{}} \implies \mathcal{Rob}. \vdash_c \varphi^{\ddot{}}[k/x_0]$
- $k \notin \mathcal{D}_{diag}^{\ddot{}} \implies \mathcal{Rob}. \vdash_c \neg \varphi^{\ddot{}}[k/x_0]$.

In order to get a contradiction, it is enough to consider the closed formula $\varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0]$ where

$\ulcorner \varphi^{\ddot{}} \urcorner$ stands for the term $\overbrace{S \dots S}^{\ulcorner \varphi^{\ddot{}} \urcorner} 0$.

Indeed, as with the Halting problem, where we asked the question whether our machine would stop on its own code as input, here we ask the question whether or not T proves $\varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0]$. This depends on whether $\ulcorner \varphi^{\ddot{}} \urcorner$ belongs to $\mathcal{D}_{diag}^{\ddot{}}$ or not.

$$\begin{aligned}
 \circ \ulcorner \varphi^{\ddot{}} \urcorner \in \mathcal{D}_{diag}^{\ddot{}} &\implies \overbrace{\mathcal{Rob}. \vdash_c \varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0]}^{\varphi^{\ddot{}} \text{ represents } \mathcal{D}_{diag}^{\ddot{}}}} \implies T \vdash_c \varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0] \implies \overbrace{\ulcorner \varphi^{\ddot{}} \urcorner \notin \mathcal{D}_{diag}^{\ddot{}}}}^{\text{by def. of } \mathcal{D}_{diag}^{\ddot{}}}. \\
 \circ \ulcorner \varphi^{\ddot{}} \urcorner \notin \mathcal{D}_{diag}^{\ddot{}} &\implies \overbrace{\mathcal{Rob}. \vdash_c \neg \varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0]}^{\varphi^{\ddot{}} \text{ represents } \mathcal{D}_{diag}^{\ddot{}}}} \implies T \vdash_c \neg \varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0] \\
 &\quad \text{since } T \supseteq \mathcal{Rob}. \\
 &\implies T \not\vdash_c \varphi^{\ddot{}}[\ulcorner \varphi^{\ddot{}} \urcorner / x_0] \implies \overbrace{\ulcorner \varphi^{\ddot{}} \urcorner \in \mathcal{D}_{diag}^{\ddot{}}}}^{\text{by def. of } \mathcal{D}_{diag}^{\ddot{}}}. \\
 &\quad \text{since } T \text{ is consistent}
 \end{aligned}$$

We obtain

$$\ulcorner \varphi^{\ddot{}} \urcorner \in \mathcal{D}_{diag}^{\ddot{}} \iff \ulcorner \varphi^{\ddot{}} \urcorner \notin \mathcal{D}_{diag}^{\ddot{}}.$$

□

^awe recall that on page 145 we defined $\mathcal{S}_{ub.}^{\mathcal{F}}(n_u, n_t, n) = \begin{cases} \ulcorner \varphi[t/x_n] \urcorner & \text{if } n_\varphi = \ulcorner \varphi \urcorner \in \mathcal{F}, n_t = \ulcorner t \urcorner \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$.

This brings to the mind what we did in the proof of Proposition 7.5.

We propose again a picture that illustrates this diagonal argument. If $(\varphi_i)_{i \in \mathbb{N}}$ is a enumeration of all the formulas with x_0 as one and only free variable, we make sure to define a formula which satisfies this requirement although it is none of them.

	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	...	φ_n	...
$\ulcorner \varphi_0 \urcorner$	0	1	1	0	1	0	...	0	...
$\ulcorner \varphi_1 \urcorner$	1	1	1	0	0	0	...	0	...
$\ulcorner \varphi_2 \urcorner$	1	0	1	0	0	0	...	1	...
$\ulcorner \varphi_3 \urcorner$	0	0	1	0	1	0	...	0	...
$\ulcorner \varphi_4 \urcorner$	0	1	0	1	1	1	...	0	...
$\ulcorner \varphi_5 \urcorner$	1	1	0	0	0	0	...	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$\ulcorner \varphi_n \urcorner$	1	0	0	0	1	1	...	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

There is a 1 on the array – for instance on row 3 and column 2 – if $T \vdash_c \varphi_2(\ulcorner \varphi_3 \urcorner)$, and there is a 0 – for instance on row 2 and column 5 – if $T \not\vdash_c \varphi_5(\ulcorner \varphi_2 \urcorner)$.

Now if T is decidable, the whole array is decidable. This means there is a Decider that on any input (n, m) accepts if there is a 1 on position (n, m) , and rejects if there is a 0. Furthermore, for the whole array is decidable, its diagonal is also decidable. Hence the complement of the diagonal is decidable as well. Finally, since all recursive sets are representable, the complement of the diagonal is represented by some formula among the enumeration – say φ_n – which inevitably stumbles on $\ulcorner \varphi_n \urcorner$.

5.4 Undecidability of first order logic

Theorem 4.1: Undecidability of first order logic

The following set is not recursive:

$$\left\{ \ulcorner \varphi \urcorner \mid \vdash_c \varphi \right\}.$$

Proof of Theorem 4.1:

Since $\mathcal{R}ob.$ is a finite $\mathcal{L}_{\mathcal{A}}$ -theory, we let $\varphi_{\mathcal{R}ob.}$ be the conjunction of the seven axioms from $\mathcal{R}ob.$. For any formula ψ we have

$$\mathcal{R}ob. \vdash_c \psi \iff \varphi_{\mathcal{R}ob.} \vdash_c \psi \iff \vdash_c \varphi_{\mathcal{R}ob.} \rightarrow \psi.$$

$$\ulcorner \psi \urcorner \in \left\{ \ulcorner \varphi \urcorner \mid \mathcal{R}ob. \vdash_c \varphi \right\} \iff \ulcorner \varphi_{\mathcal{R}ob.} \rightarrow \psi \urcorner \in \left\{ \ulcorner \varphi \urcorner \mid \vdash_c \varphi \right\}.$$

Therefore, if the set of codes of universally valid formulas were decidable, then Robinson arithmetic would also be decidable. □

5.5 Gödel's 1st incompleteness theorem

Everything is now ready to prove the first incompleteness theorem in a couple of swift moves.

Theorem 5.1: Gödel's 1st incompleteness theorem

If $T \supseteq \mathcal{R}ob.$ is any recursive $\mathcal{L}_{\mathcal{A}}$ -theory, then

T is incomplete.

Proof of Theorem 5.1:

We prove the contrapositive. If T is complete, then T is consistent. By Theorem 3.2, T is then undecidable. Corollary 3.1 states that every theory which is both recursive and complete is decidable. Therefore, T is not recursive.

□

Another way of stating this theorem (very common among philosophers) is the following:

There exists a true sentence that is not provable.

Or even,

There exists a sentence that is true although it is not provable.

“Provable” usually refers to Peano Arithmetic, and “true” means true in the standard model. And very often people add

“Taking this true sentence as an axiom inevitably yields another one that is not provable.”

Of course the understatement hidden behind this assertion is that “the complete theory of the standard model is not recursive”.

To put it in our language, what these philosophers do is they consider some theory $T \supseteq \mathcal{R}ob$. (usually they take *Peano Arithmetic*). This theory being both consistent (since they presume it is satisfied inside the standard model $\mathbb{N}$) and recursive (that criterion is often omitted), it is incomplete by Gödel’s 1st incompleteness theorem, there exists some closed formula φ such that both $T \not\vdash_c \varphi$ and $T \not\vdash_c \neg\varphi$. If $\mathbb{N} \models \varphi$, then φ is the formula they have in mind, and if $\mathbb{N} \not\models \varphi$, then $\neg\varphi$ is the formula they have in mind.

The assumption “*taking this true sentence as an axiom inevitably yields another one that is not provable*” simply refers to the fact that for any closed formula φ such that $\mathbb{N} \models \varphi$ holds, and any recursive theory T such that $\mathbb{N} \models T$ holds as well, we have

(1) $\mathbb{N} \models T \cup \{\varphi\}$, hence $T \cup \{\varphi\}$ is consistent,

(2) $T \cup \{\varphi\}$ remains recursive.

Therefore, $T \cup \{\varphi\}$ is another incomplete theory, which yields another formula ψ such that $T \not\vdash_c \psi$ and $T \not\vdash_c \neg\psi$ but $\mathbb{N} \models \psi$.